Transport Phenomena

1. Boltzmann Equation and Relaxation Time
2. Electrical Conductivity of Metals
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Boltzmann Equation and Relaxation Time

Transport phenomena, such as the flow of electric current in solids, involve two characteristic mechanisms with opposite effects: the driving force of the external fields and the dissipative effect of the scattering of the carriers by phonons and defects. The interplay between the two mechanisms is described by the Boltzmann equation. In thermal equilibrium,

$$f_0[E(\boldsymbol{k})] = \frac{1}{e^{[E(\boldsymbol{k})-E_F]/\not{k}T} + 1} \ .$$

This equilibrium distribution f_0 is independent of ***r*** because of the assumed homogeneity.

Away from equilibrium, where we merely assume local equilibrium over regions large compared with atomic dimensions, the required distribution $f(\boldsymbol{r},\boldsymbol{k},t)$ can be both space and time dependent. Under an applied external field $\mathcal{E}$, an electron that is at ***r*** and ***k*** at time *t*, will have had the coordinates $\boldsymbol{r} - \boldsymbol{v}(\boldsymbol{k})dt$ and $\boldsymbol{k} - (-e)\mathcal{E}dt/\hbar$ at time *t-dt*. In the absence of collisions, each electron with coordinates ***r*** - *vdt* and ***k*** + $e\mathcal{E}dt/\hbar$ at *t-dt* must arrive at ***r***, ***k*** at time t,

$$f(\boldsymbol{r},\boldsymbol{k},t) = f(\boldsymbol{r} - v\,dt, \boldsymbol{k} + e\mathcal{E}dt/\hbar, t - dt) \ .$$

If we express the change in *f* due to scattering by the term $(\partial f/\partial t)_{\mathrm{s}}$, then the correct equation is

$$f(\boldsymbol{r},\boldsymbol{k},t) = f(\boldsymbol{r}-\boldsymbol{v}\,dt,\boldsymbol{k}+e\mathscr{E}\,dt/\hbar,t-dt) + \left(\frac{\partial f}{\partial t}\right)_{\mathrm{s}} dt\ .$$

Expanding this up to terms linear in *dt* gives the Boltzmann equation,

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f - \frac{e}{\hbar}\mathscr{E}\cdot\nabla_{\boldsymbol{k}} f = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{s}}}\ .$$

Probability for transitions from the Bloch state $\psi_{\boldsymbol{k}}$ to $\psi_{\boldsymbol{k}'}$ is $w_{\boldsymbol{k}'\boldsymbol{k}} \propto |\langle \boldsymbol{k}'|\mathscr{H}'|\boldsymbol{k}\rangle|^2$

$$\left(\frac{\partial f(\boldsymbol{k})}{\partial t}\right)_{\mathrm{s}} = \frac{V}{(2\pi)^3}\int d\boldsymbol{k}'\{[1-f(\boldsymbol{k})]w_{\boldsymbol{k}\boldsymbol{k}'}f(\boldsymbol{k}') - [1-f(\boldsymbol{k}')]w_{\boldsymbol{k}'\boldsymbol{k}}f(\boldsymbol{k})\}\ .$$

We employ the *relaxation time ansatz*, assuming that the rate at which *f* returns to the equilibrium distribution f_0 due to scattering is proportional to the deviation of *f* from f_0, then

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{s}} = -\frac{f(\boldsymbol{k}) - f_0(\boldsymbol{k})}{\tau(\boldsymbol{k})}$$

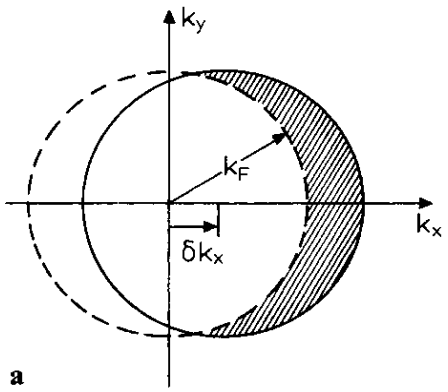
If f does not depend on position (i.e., $\nabla_r f=0$), then under the influence of an electric field $\mathcal{E}$, it follows that the stationary state ($\partial f/\partial t=0$) is given by

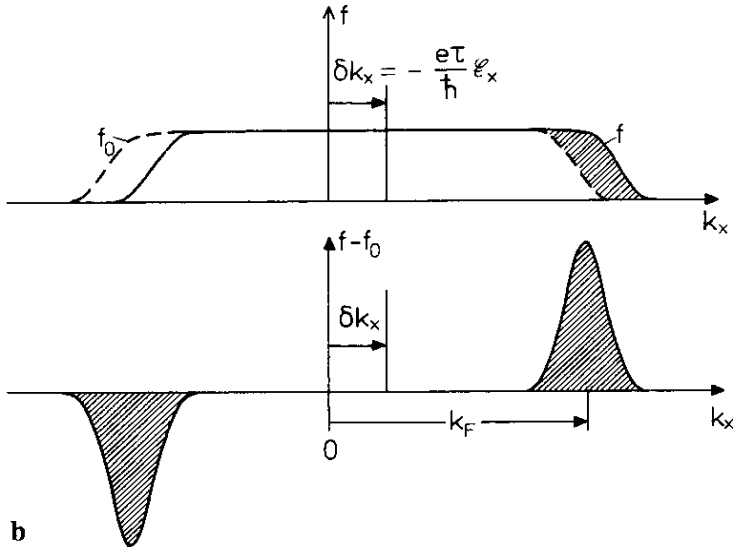
$$-\frac{e}{\hbar}\mathscr{E}\cdot\nabla_k f=-[f(\boldsymbol{k})-f_0(\boldsymbol{k})]/\tau(\boldsymbol{k})\ ,$$

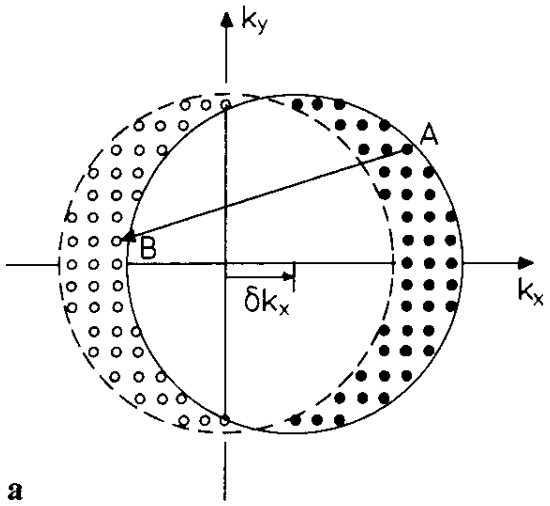
$$f(\boldsymbol{k})=f_0(\boldsymbol{k})+\frac{e}{\hbar}\tau(\boldsymbol{k})\mathscr{E}\cdot\nabla_k f(\boldsymbol{k})\ .$$

To the first order in $\mathcal{E}$,

$$f(\boldsymbol{k})\simeq f_0(\boldsymbol{k})+\frac{e}{\hbar}\tau(\boldsymbol{k})\mathscr{E}\cdot\nabla_k f_0(\boldsymbol{k})\simeq f_0\left(\boldsymbol{k}+\frac{e}{\hbar}\tau(\boldsymbol{k})\mathscr{E}\right)\ .$$

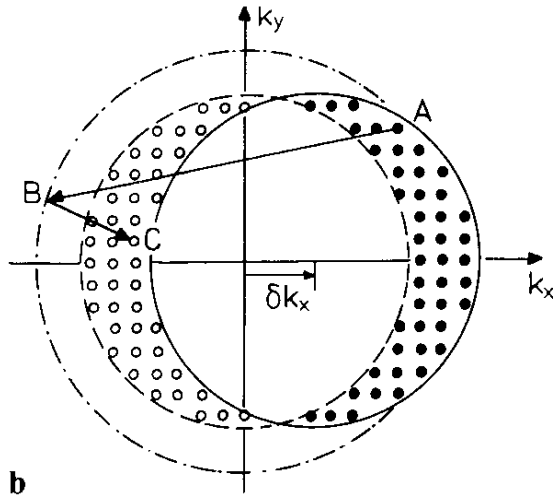






a

The stationary state of the distribution is represented as a displaced Fermi sphere in **a** (full line). If the external field is switched off, the displaced sphere relaxes back to the equilibrium distribution (dashed line).



b

Only inelastic scattering (**a**) can cause this return to equilibrium. Elastic collisions alone, e.g. from defects, cannot facilitate the return to equilibrium. Without inelastic collisions the Fermi sphere would merely expand (**b**).

Electrical Conductivity of Metals

Drude Model: an ideal electron gas in the solid,

$$m\dot{v} + \frac{m}{\tau} v_{\mathrm{D}} = -e\mathscr{E} \ .$$

The scattering is accounted for by the friction term mv_{D}/τ where $v_{\mathrm{D}} = v - v_{\mathrm{therm}}$ is the so-called drift velocity, i.e., the additional velocity due to the field, over and above the thermal velocity. For the stationary case ($\dot{v} = 0$) one has $v_{\mathrm{D}} = -\frac{e\tau}{m}\mathscr{E}$, and hence the current density

$$j = -env_{\mathrm{D}} = ne\mu\mathscr{E} = \frac{e^2\tau n}{m}\mathscr{E} \ .$$

The electrical conductivity σ and mobility μ are therefore

$$\sigma = j/\mathscr{E} = \frac{e^2 n\tau}{m} \qquad \text{and} \qquad \mu = \frac{e\tau}{m} \ .$$

Note that, in this simple model, all free electrons contribute to the current. This view is in contradiction to the Pauli principle.

Semiclassical Approach: dynamics of band electrons are considered.

The contribution to the current of electrons in the volume element ***dk*** is

$$\boldsymbol{j} = -\frac{e}{8\pi^3}\int d\boldsymbol{k}\, \boldsymbol{v}(\boldsymbol{k}) f(\boldsymbol{k}) \ ,$$ ***f*(*k*)** is the occupation probability function.

For an electric field $\mathscr{E}_x$ in the *x*-direction, the electrical current density is

$$\boldsymbol{j} = -\frac{e}{8\pi^3}\int d\boldsymbol{k}\, \boldsymbol{v}(\boldsymbol{k}) \left[f_0(\boldsymbol{k}) + \frac{e\tau(\boldsymbol{k})}{\hbar}\mathscr{E}_x \frac{\partial f_0}{\partial k_x} \right]$$

Since the integral is over the whole Brillouin zone and $f_0(\boldsymbol{k})$ has inversion symmetry about $\boldsymbol{k} = 0$, the integral over $v_x\, f_0$ vanishes. Furthermore, since

$$\frac{\partial f_0}{\partial k_x} = \frac{\partial f_0}{\partial E}\hbar v_x \ , \quad \text{and} \quad j_x = -\frac{e^2}{8\pi^3}\mathscr{E}_x \int d\boldsymbol{k}\, v_x^2(\boldsymbol{k})\tau(\boldsymbol{k})\frac{\partial f_0}{\partial E} \ .$$

The specific electrical conductivity is therefore

$$\sigma = j_x / \mathscr{E}_x = -\frac{e^2}{8\pi^3}\int d\boldsymbol{k}\, v_x^2(\boldsymbol{k})\tau(\boldsymbol{k})\frac{\partial f_0}{\partial E} \ .$$

The energy region over which the Fermi function $f_0(E)$ changes rapidly has a width of about 4 kT. It also has inversion symmetry about the point (E_F, f_0 (E_F) = 1/2). Thus, to a good approximation,

$$\frac{\partial f_0}{\partial E} \approx -\delta(E - E_F) \ . \quad \text{With} \quad d\boldsymbol{k} = df_E dk_\perp = df_E \frac{dE}{|\nabla_{\boldsymbol{k}} E|} = df_E \frac{dE}{\hbar v(\boldsymbol{k})} \ ,$$

$$\sigma \simeq \frac{e^2}{8\pi^3\hbar} \int df_E dE \frac{v_x^2(\boldsymbol{k})}{v(\boldsymbol{k})} \tau(\boldsymbol{k}) \delta(E - E_F) \simeq \frac{e^2}{8\pi^3\hbar} \int\limits_{E=E_F} \frac{v_x^2(\boldsymbol{k})}{v(\boldsymbol{k})} \tau(\boldsymbol{k}) df_E$$

The electrical conductivity σ of a metal can thus be expressed as a *surface integral over the Fermi surface* $E(\boldsymbol{k})=E_F$ in $\boldsymbol{k}$-space. Only the velocity $v(E_F)$ and the relaxation time $\tau(E_F)$ of the electrons at the Fermi surface appear in the microscopic description. The above equation expresses precisely the fact that only electrons in the vicinity of the Fermi energy are relevant for current transport in a metal, as expected from the Pauli exclusion principle.

For electrons in an exactly parabolic band (quasi-free electrons) we have

$$v(E_F) = \hbar k_F / m^* \qquad \text{and} \qquad \int\limits_{E_F} df_E = 2(4\pi k_F^2) \ .$$

If $k_B T \ll E_F$, we have $n = \frac{2(4/3)\pi k_F^3}{8\pi^3}$, i.e., $k_F^3 = 3\pi^2 n$.

Thus, the electrical conductivity σ and mobility μ are given by

$$\sigma \simeq \frac{e^2 \tau(E_F)}{m^*} n \quad \text{and} \quad \mu \simeq \frac{e\tau(E_F)}{m^*} .$$

To understand the temperature dependence of the resistance of metals, it suffices to consider the temperature dependence of $\tau(E_F)$ or μ, because the electron concentration n is independent of temperature. Assuming that the *phonon* and *defect* scatterings are independent of one another, the total scattering probability is the sum of the individual scattering probabilities. The scattering probability is inversely proportional to the relaxation time. It therefore follows that

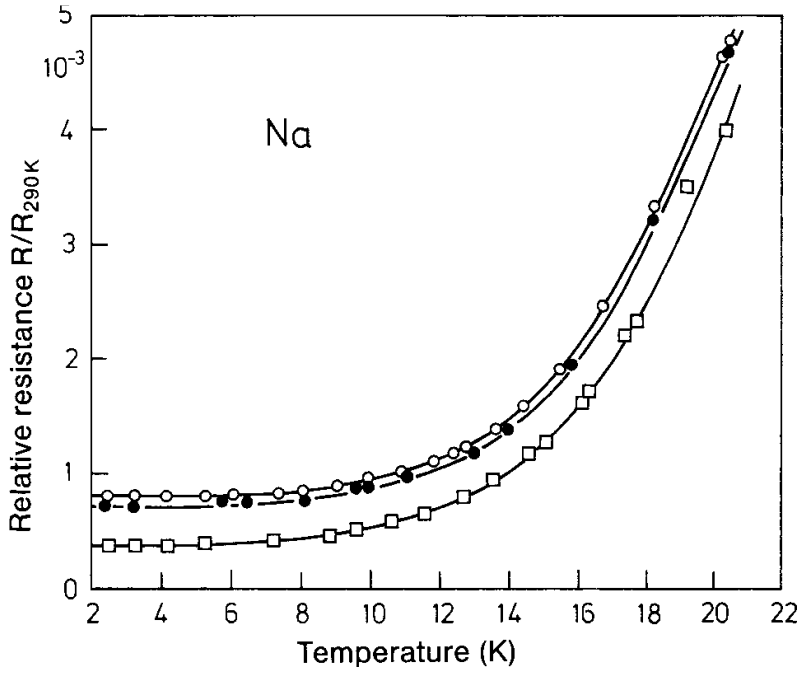
$$\frac{1}{\tau} = \frac{1}{\tau_{ph}} + \frac{1}{\tau_{def}}$$

τ_{def} is usually temperature independent and τ_{ph} is proportional to the mean square vibrational amplitude $\langle u^2(\boldsymbol{q}) \rangle$, so

$$\frac{1}{\tau_{ph}} \sim \langle u^2(\boldsymbol{q}) \rangle \sim \frac{k_B T}{M\omega_{\boldsymbol{q}}^2} .$$

We can write the resistivity $\rho = 1/\sigma \propto 1/\tau$ of a metal as the sum of a temperature-independent residual resistivity ρ_{def} (due to defects) and a part due to phonon scattering $\rho_{ph}(T)$ which is linear in temperature at high temperature:

$$\varrho = \varrho_{\mathrm{ph}}(T) + \varrho_{\mathrm{def}} \cdot$$



Left figure shows the experimentally measured electrical resistance of Na at low temperature. Below about 8 K, a temperature-independent ρ_{def} residual resistance is observed, which depends on the defect concentration of the sample. At higher temperatures, the component described by the Gruneisen formula becomes evident, and above 18 K ρ_{ph} displays the linear de pendence $\rho_{ph} \sim T$.

Motion of Electrons in Bands

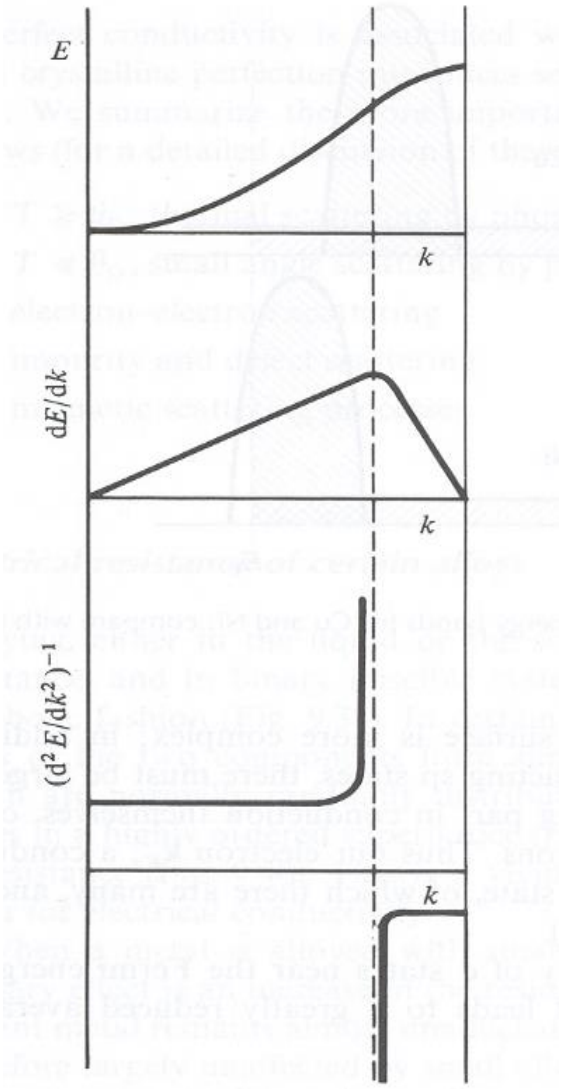
Motion of equation for an electron in a crystal is

$$\hbar \frac{d\mathbf{k}}{dt} = \mathbf{F}$$

The effective mass defined as

$$\frac{1}{m^*} \equiv \frac{1}{\hbar^2} \frac{\partial^2 E}{\partial k_i \partial k_j} \qquad i,\ j = x,\ y,\ z$$

is useful because it allows us to retain the notion of a free-electron even when we have a periodic potential, as long as we use m* to account for the effect of the lattice on the acceleration of the electron.



Scattering of Electrons in Bands

An exactly periodic lattice of positive cores does not cause scattering.
Perturbations of the stationary Bloch states can only occur in two ways:

(I) Within the one-electron approximation, where interactions between electrons are neglected, the only sources of electron scattering are deviations from strict periodicity in the lattice. These may be:

a) defects in the lattice that are fixed in time and space such as vacancies, dislocations, impurities, etc.

b) deviations from periodicity that vary in time, i.e., lattice vibrations.

(II) The one-electron approximation neglects interactions between electrons. Electron-electron collisions, which are not contained in the concept of a non-interacting Fermi gas, can in fact perturb the stationary Bloch states. As we will see, this effect is usually much less significant than those noted in (I).

The decisive quantity for the description of an electron scattering process is the probability $w_{k'k}$ that the electron will be scattered from a Bloch state $\psi_{\boldsymbol{k}}(\boldsymbol{r})$ to a state $\psi_{\boldsymbol{k}'}(\boldsymbol{r})$ under the influence of one of the previously described imperfections.

$$w_{\boldsymbol{k}'\boldsymbol{k}} \sim |\langle \boldsymbol{k}'|\mathscr{H}'|\boldsymbol{k}\rangle|^2 = |\int d\boldsymbol{r}\psi^*_{\boldsymbol{k}'}(\boldsymbol{r})\mathscr{H}'\psi_{\boldsymbol{k}}(\boldsymbol{r})|^2 ,$$

where $\mathscr{H}'(\boldsymbol{r})$ is the perturbation to the Hamiltonian.

If $\mathscr{H}'(\boldsymbol{r})$ is a potential that is constant in time, such as that of a static defect, then we expect only *elastic* scattering of the Bloch waves with conservation of energy.

If $\mathscr{H}'(\boldsymbol{r},t)$ is a potential that varies in time, as appropriate for the perturbation due to a lattice wave (phonon), then the scattering is *inelastic*.

Energy conservation also applies to the scattering of conduction electrons by phonons:

$$E(\boldsymbol{k'}) - E(\boldsymbol{k}) = \pm\hbar\omega(\boldsymbol{q}) .$$

For scattering by a phonon with wave vector $\boldsymbol{q}$, the perturbation potential $\mathscr{H}'$ naturally has a spatial dependence exp (i$\boldsymbol{q}\cdot\boldsymbol{r}$). This means

$$\langle \boldsymbol{k}'|\mathrm{e}^{\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{r}}|\boldsymbol{k}\rangle = \int d\boldsymbol{r} u^{*}_{\boldsymbol{k}'} u_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{r}} \quad , \quad \psi_{\boldsymbol{k}}(\boldsymbol{r}) = u_{\boldsymbol{k}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

Because ($u^{*}_{\boldsymbol{k}}$,$u_{\boldsymbol{k}}$) has the periodicity of the lattice and can be expanded as a Fourier series in terms of reciprocal lattice vectors, the matrix element above is non-zero only when

$$\boldsymbol{k}' - \boldsymbol{k} = \boldsymbol{q} + \boldsymbol{G}\ .$$

If we take *energy conservation* and ***k**-conservation* together, then, we see that scattering of Bloch-state electrons, formally should be represented with a wavepacket, can be well described in the particle picture.

Electron-Electron Scattering

For a collision between two electrons (1)+(2) → (3)+(4), we must have

$E_1 + E_2 = E_3 + E_4$, where $E_i = E(\boldsymbol{k}_i)$ denotes the one-particle energy of an electron in a non-interacting Fermi gas. Furthermore for the corresponding k-vectors: $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 + \boldsymbol{G}$.

Let us assume that one electron occupies the state $E_1 > E_F$, an excited state just above the Fermi level; the second electron involved in the collision is inside the Fermi sphere with $E_2 < E_F$. For scattering to states E_3 and E_4, the Pauli principle demands that E_3 and E_4 must be unoccupied. Thus,

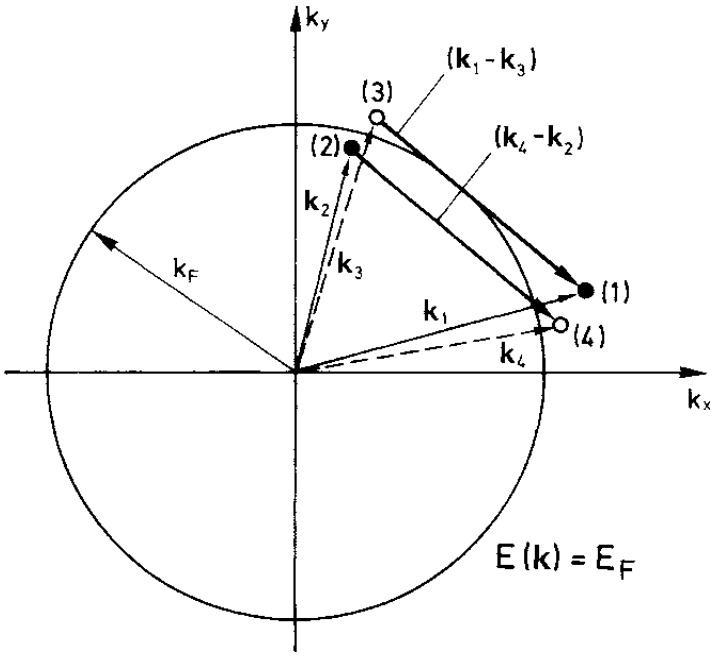
$$E_1 > E_F\,,\ E_2 < E_F\,,\ E_3 > E_F\,,\ E_4 > E_F\ .$$

From energy conservation, it then follows that

$$E_1 + E_2 = E_3 + E_4 > 2E_F \qquad \text{and} \qquad (E_1 - E_F) + (E_2 - E_F) > 0\ .$$

If $(E_1 - E_F) < \varepsilon_1$, then $|E_2 - E_F| = |\varepsilon_2| < \varepsilon_1$

The thermal broadening of the Fermi function is of the order $k_B T$, so that the final state E_1 must lie within this energy of E_F , i.e. $\varepsilon_1 \sim k_B T$.



Since only the fraction ~ε_1/E_F of all electrons may scatter with the electron in the state E_1. If E_1 and E_2 are in the shell $\pm\varepsilon_1$ around E_F, then because of $\boldsymbol{k}$ conservation, E_3 and E_4 must also lie in the shell $\pm\varepsilon_1$ around E_F. The $\boldsymbol{k}$ conservation in the form $\boldsymbol{k}_1 - \boldsymbol{k}_3 = \boldsymbol{k}_4 - \boldsymbol{k}_2$ means that the connecting lines (1)−(3) and (2)−(4) in the figure must be equal. Because only a fraction ~ε_1/E_F of all unoccupied states are allowed final states, the Pauli principle further reduces the scattering probability by a factor ε_1/E_F .

Let us assume that the cross-section for the scattering of an electron from a defect in the lattice is of the order Σ_0, then the cross-section for the *e*-*e* scattering Σ is $\Sigma \propto \left(\frac{k_B T}{E_F}\right)^2 \Sigma_0$, and typically, k_B/E_F ~10^{-5} K^{-1}.

The probability of electron-electron scattering at a temperature of 1 K is about a factor ~10^{-10} smaller than that of electron-defect scattering.

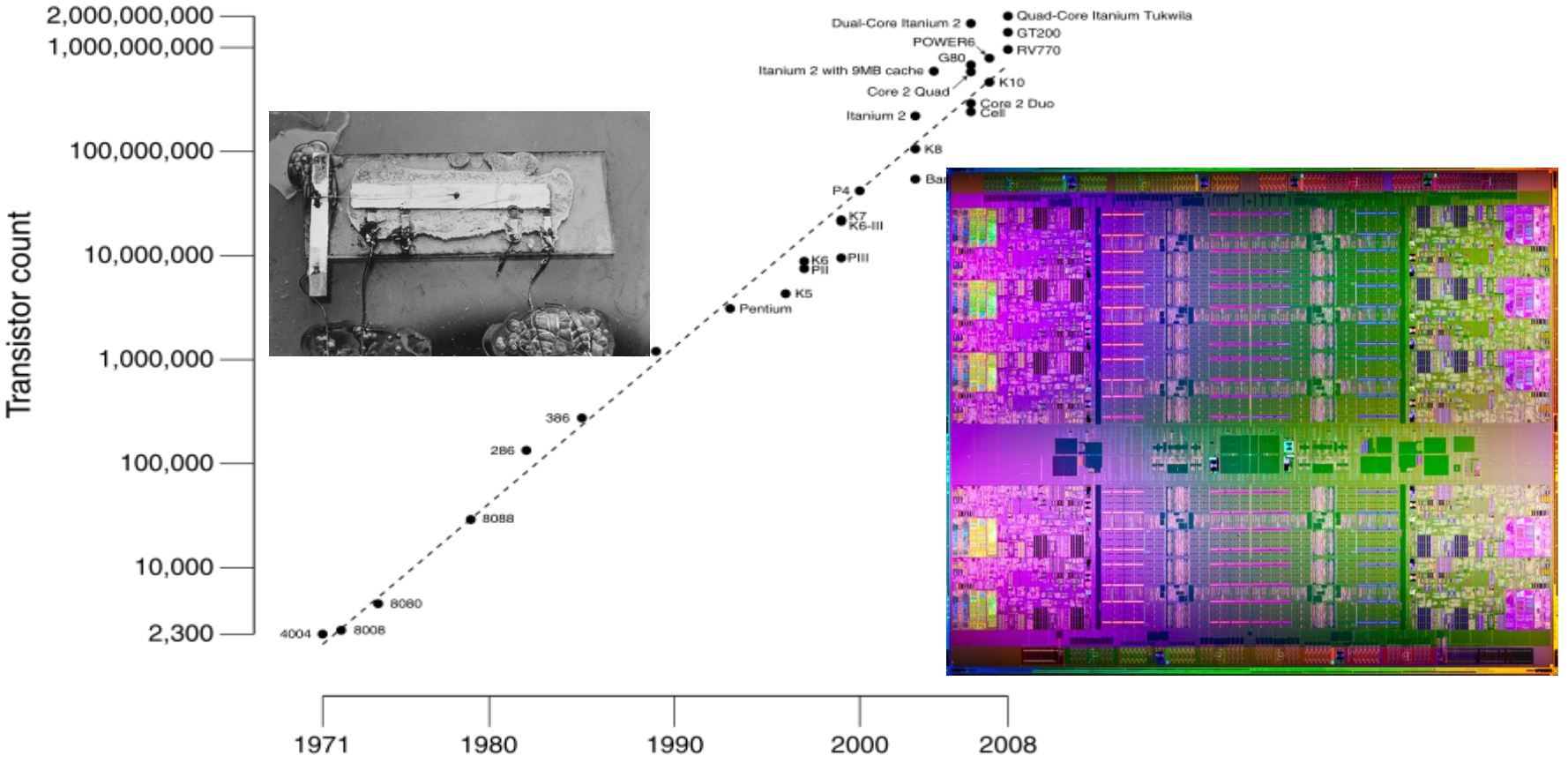
Quantum Transport in Nanostructures

- Small number of states can affect the overall current
- Wavefunction coherence lengths are comparable to characteristic device dimensions
- Single electrons charging effects can be significant

These can amount to overall macroscopic electronic properties that show deviations from bulk electronic properties.

Moore's Law

The number of transistors per microchip doubles roughly every two years.



Important Length Scales

Various characteristic length scales of a propagating electron:

Elastic mean free path (l_e): average distance the electrons travel without being elastically scattered

$l_e = v_F \tau_e$. v_F denotes the Fermi velocity of the electrons

Inelastic mean free path (l_{in}): average distance the electrons travel before their energy is changed, mainly due to interaction with phonons

$l_{in} = v_F \tau_{in}$. τ_{in} denotes the mean time between inelastic scattering events

Phase coherent length (l_Φ): average distance the electrons travel before their phase is randomized

$l_\Phi = v_F \tau_\Phi$. τ_Φ denotes the dephasing time of the electrons

Fermi wavelength (λ_F): de Broglie wavelength of Fermi electrons

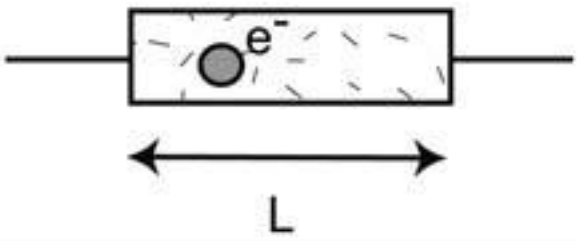
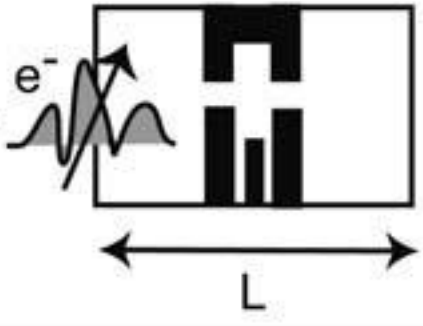
in d = 3: $\lambda_F = 2^{3/2}(\pi/3n)^{1/3}$

in d = 2: $\lambda_F = (2\pi/n)^{1/2}$

in d = 1: $\lambda_F = 4/n$

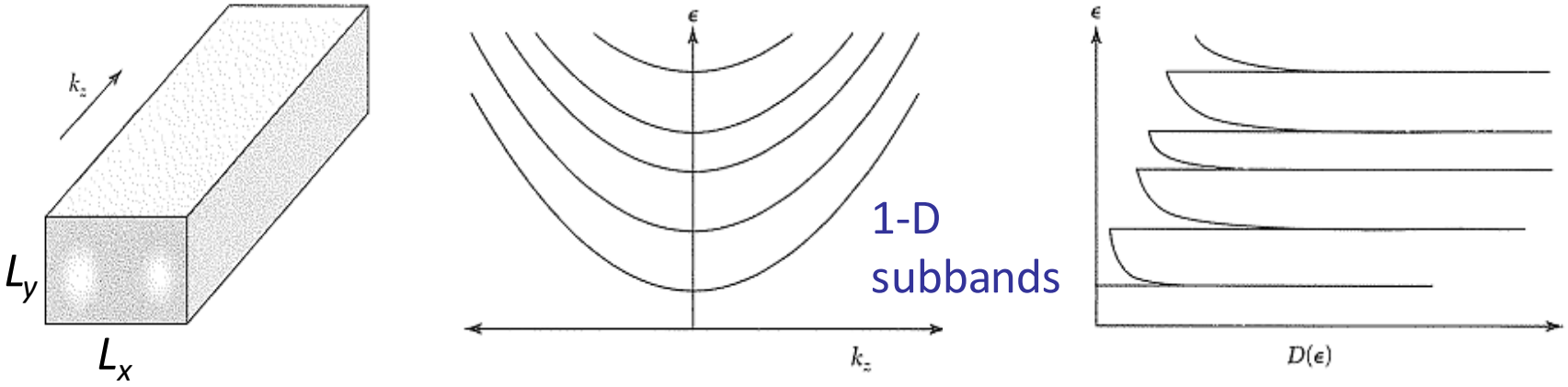
For metals, λ_F is below 1nm and for quasi-metallic 2D electron gases (2DEG), λ_F lies typically between 50 and 100nm.

Important Mesoscopic Regimes

conventional device: 		mesoscopic device: 	
$L \gg l_e$	diffusive	$L \lesssim l_e$	ballistic
$L \gg l_\phi$	incoherent	$L \lesssim l_\phi$	phase coherent
$L \gg \lambda_F$	no size quantization	$L \lesssim \lambda_F$	size quantization
$e^2/C < k_B\Theta$	no single electron charging	$e^2/C \gtrsim k_B\Theta$	single electron charging effects
$L \gg l_s$	no spin effects	$L \lesssim l_s$	spin effects

Electronic Structure of 1-D Systems

Consider a quasi one-dimensional wire with a diameter of the order of the Fermi-wavelength λ_F and a length $L_z << l_e, l_\Phi$. Electron transport through the wire is called ballistic in that case since an electron passes from one end to the other without being scattered at all.



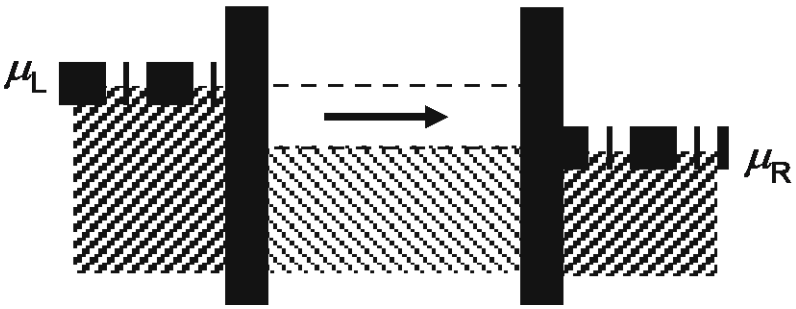
For free electrons the energy in the subbands is

$$E_i = \varepsilon_i(k_x, k_y) + \frac{\hbar^2 k_z^2}{2m^*}, \qquad \psi(x,y,z) = \psi_{i,j}(x,y)e^{ikz}$$

i, j = quantum numbers in the cross section

Thus, the energy bands represent a sequence of parabolas, each of which describes one transport channel shown in the above figure.

Electrical Transport in 1-D



The difference between the chemical potentials is given by the voltage U between the contacts,

$$\mu_{\mathrm{L}} - \mu_{\mathrm{R}} = eU.$$

Only the electron states between μ_L and μ_R contribute to the current flow from left to right. The current in one subband *i* is then

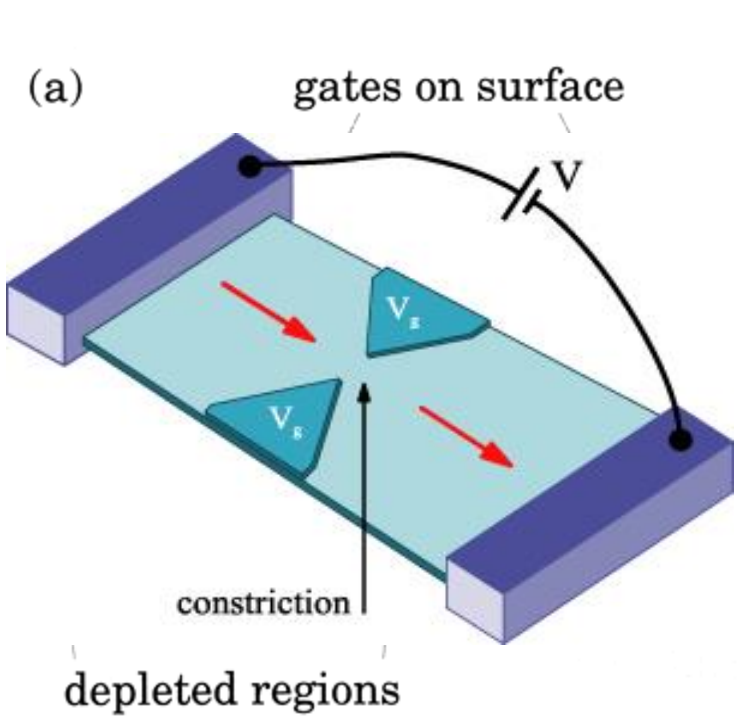
$$I_i = e \int_{\mu_R}^{\mu_L} D_i^{(1)}(E)\, \boldsymbol{v}_i(E)\, \mathrm{d}E$$

Since $D^{(1)}(E) = \frac{1}{\pi \hbar\, \mathrm{d}E/\mathrm{d}k_z}$ and $v_i(E) = \frac{1}{\hbar}\frac{\mathrm{d}E_i}{\mathrm{d}k_z}$ $\Rightarrow$ $I_i = \frac{2e^2}{h} U$.

The universal conductance is $G_0 = \frac{2e^2}{h} = 7.74809 \times 10^{-5}\ \Omega^{-1} = 1/12906.4\ \Omega$.

If more than one channel fits into the conductor then each channel contributes $2e^2/h$ to the total conductance. The conductance quantum reduces to half the value e^2/h if the spin-degeneracy is lifted by a strong magnetic field.

Conductance of Quantum Point Contact (QPC)



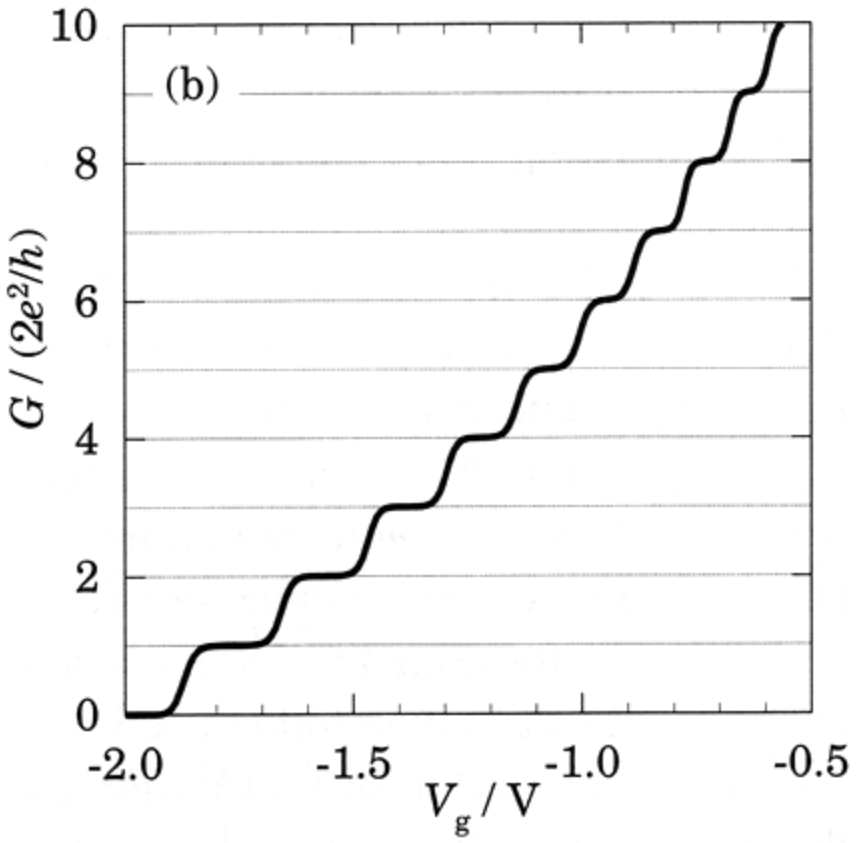


FIGURE 5.22. (a) Layout of a typical quantum point contact, a short constriction defined by patterned metal gates on the surface of a heterostructure containing a 2DEG. (b) Calculated conductance $G(V_g)$ as a function of gate voltage V_g. [From Nixon, Davies, and Baranger (1991).]

Electron Flow Close to a QPC

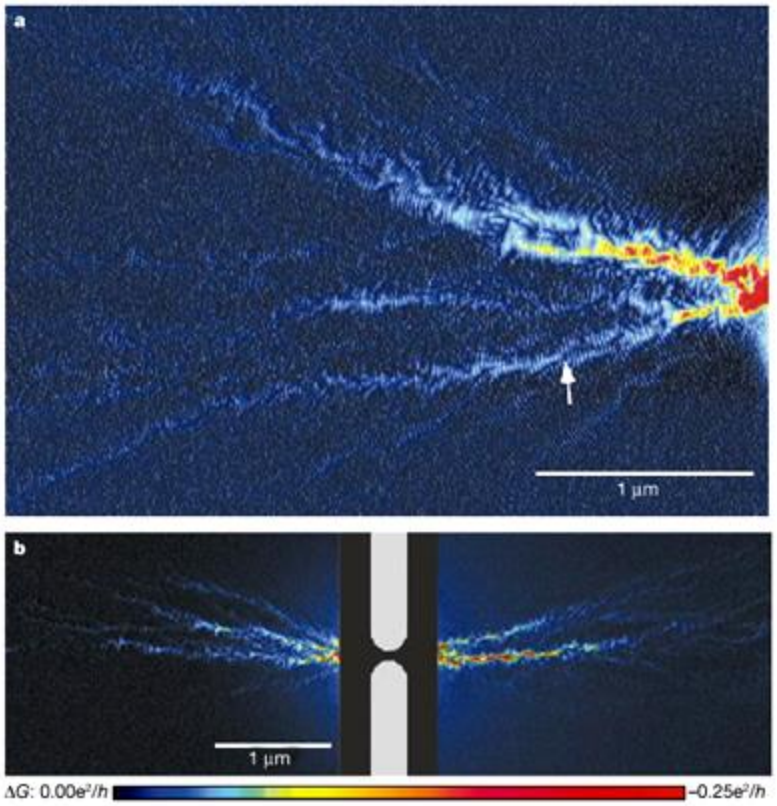
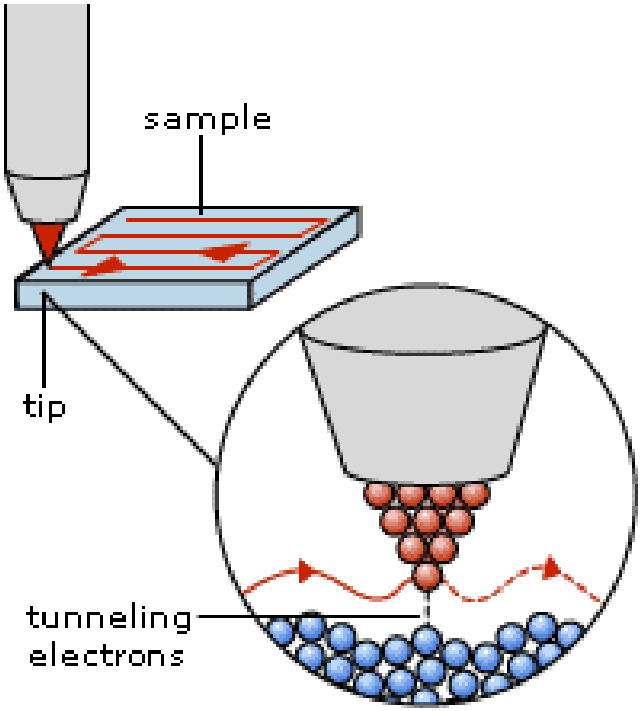
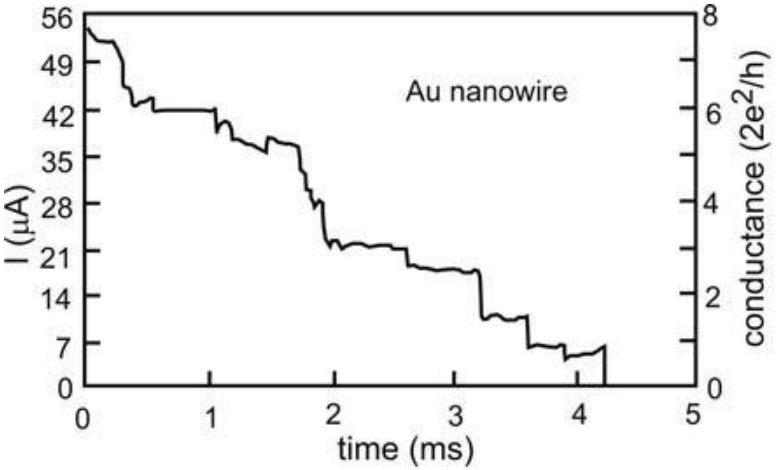


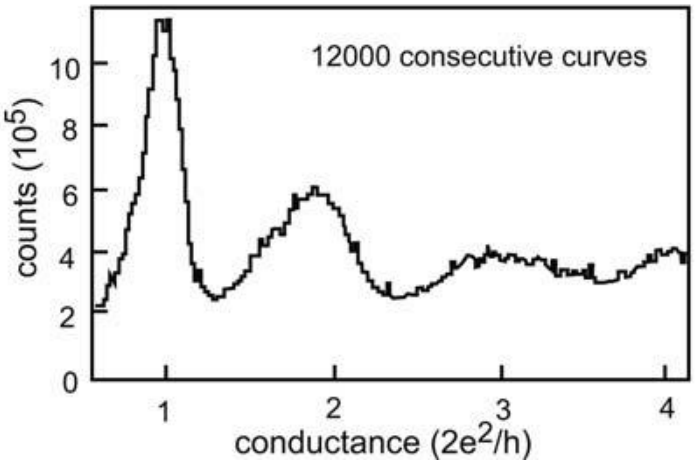
Figure 2 Experimental images of electron flow. **a**, Image of electron flow from one side of a QPC at $T = 1.7$ K, biased on the $G = 2e^2/h$ conductance step. Dark regions correspond to areas where the tip had little effect on QPC conductance, and hence are areas of low electron flow. The colour varies and the height in the scan increases with increasing electron flow. Narrow branching channels of electron flow are visible, and fringes spaced by $\lambda_F/2$, half the Fermi wavelength, are seen to persist across the entire scan. **b**, Images of electron flow from both sides of a different QPC, again biased on the $G = 2e^2/h$ conductance step. The gated region in the centre was not scanned. Strong channelling and branching are again clearly visible. The white arrow points out one example of the formation of a cusp downstream from a dip in the potential.

- Electrons are wave with wavevector $\boldsymbol{k}_F$
- Interference stripe with $\lambda = \dfrac{1}{2k_F}$

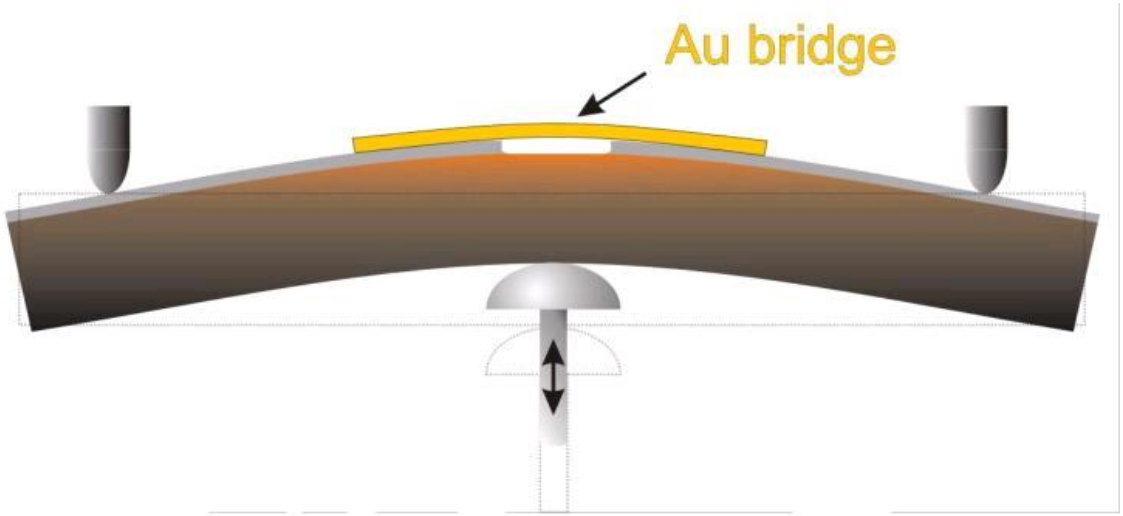
QPC Formed in STM

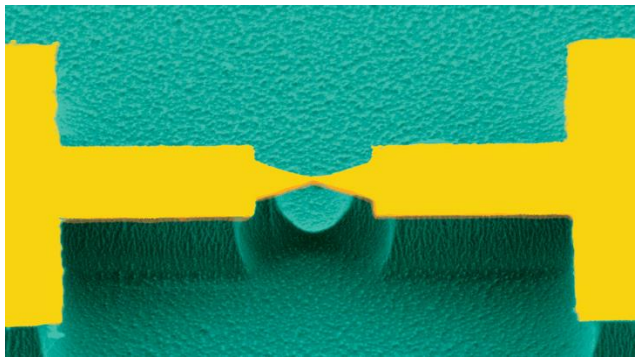






Molecular Break Junctions





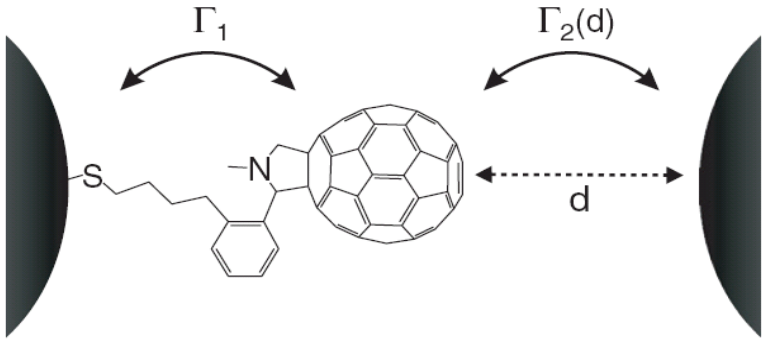


Figure 1. Schematic representation of a break junction with a thiolated C_{60} molecule anchored to the left electrode. The distance *d* between the molecule and the right electrode can be adjusted by opening and closing the junction.

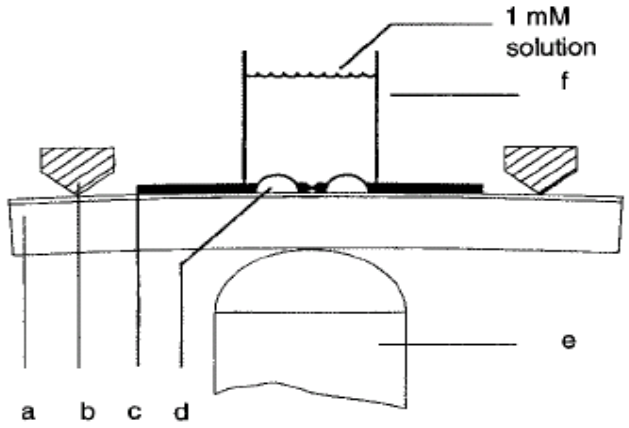
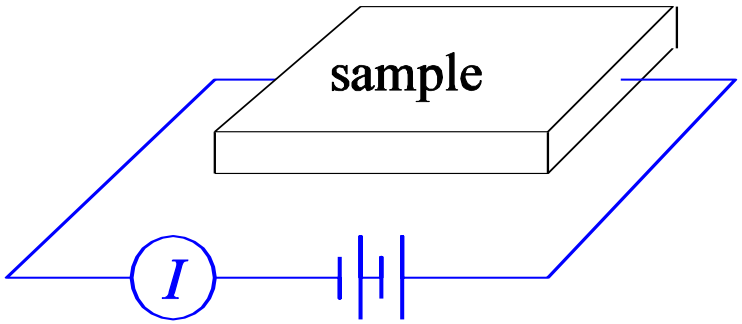
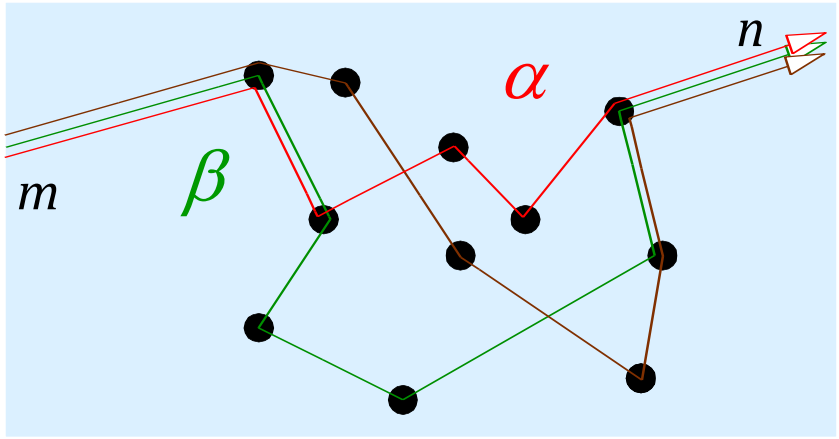


Fig. 1. A schematic of the MCB junction with (a) the bending beam, (b) the counter supports, (c) the notched gold wire, (d) the glue contacts, (e) the pizeo element, and (f) the glass tube containing the solution.

Quantum Interference





$t_{nm,\alpha}$, $t_{nm,\beta}$: amplitude for transmission along paths α, β

$$g = \sum_{m,n} |t_{nm}|^2$$

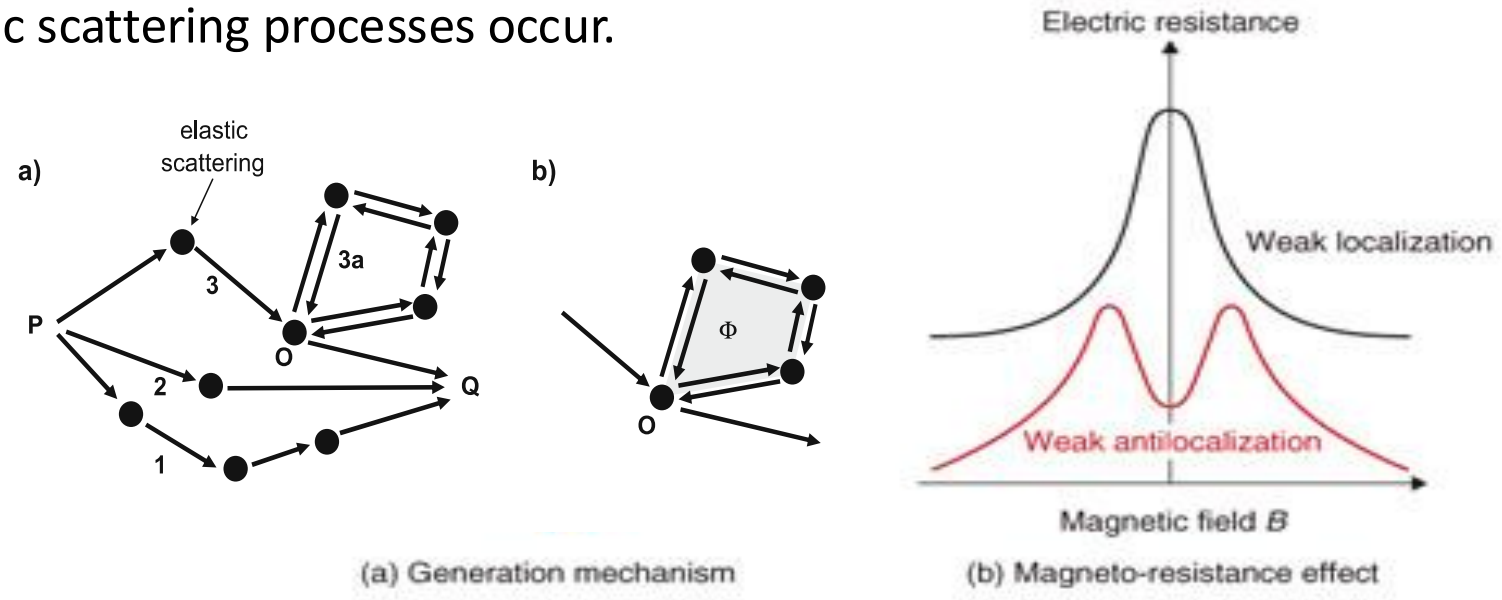
$$= \sum_{m,n} \sum_{\alpha} |t_{nm,\alpha}|^2 + \sum_{m,n} \sum_{\alpha \neq \beta} t_{nm,\alpha} (t_{nm,\beta})^*$$

$$= g_{\text{class}} + \delta g$$

In general: δg small, random sign

Weak Localization

Weak localization is a physical effect which occurs in disordered electronic systems at very low temperatures. Diffusive transport from P to Q in which elastic scattering processes occur.

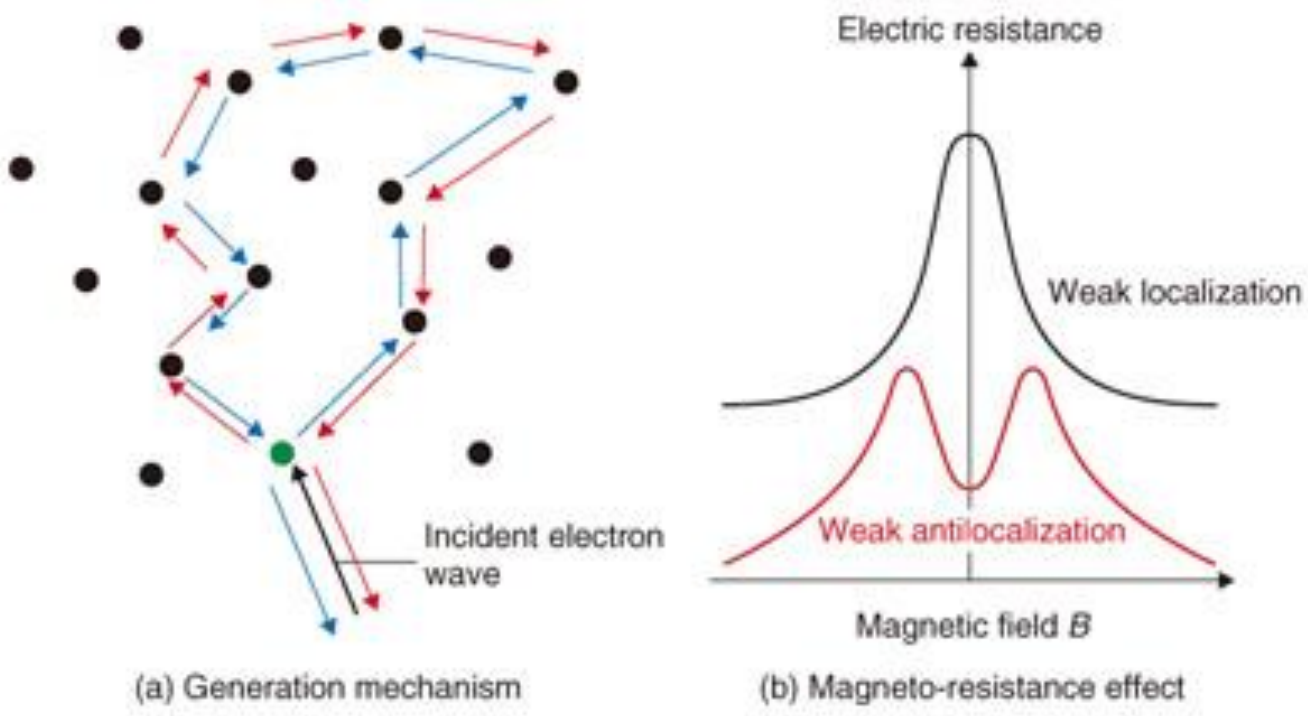


The trajectory 3a describes a closed loop that can be circulated about clockwise and counter clockwise. Constructive interference may occur which leads again to an enlarged backscattering that entails an enhancement of the resistance, i.e.,

$$A^2 = (A_1 + A_2)^2 = A_1^2 + A_2^2 + 2\,A_1A_2 = 4\,A_1^2$$

Interference effects double the classical contribution and (slightly) suppress the conductance. This is called *weak localization*.

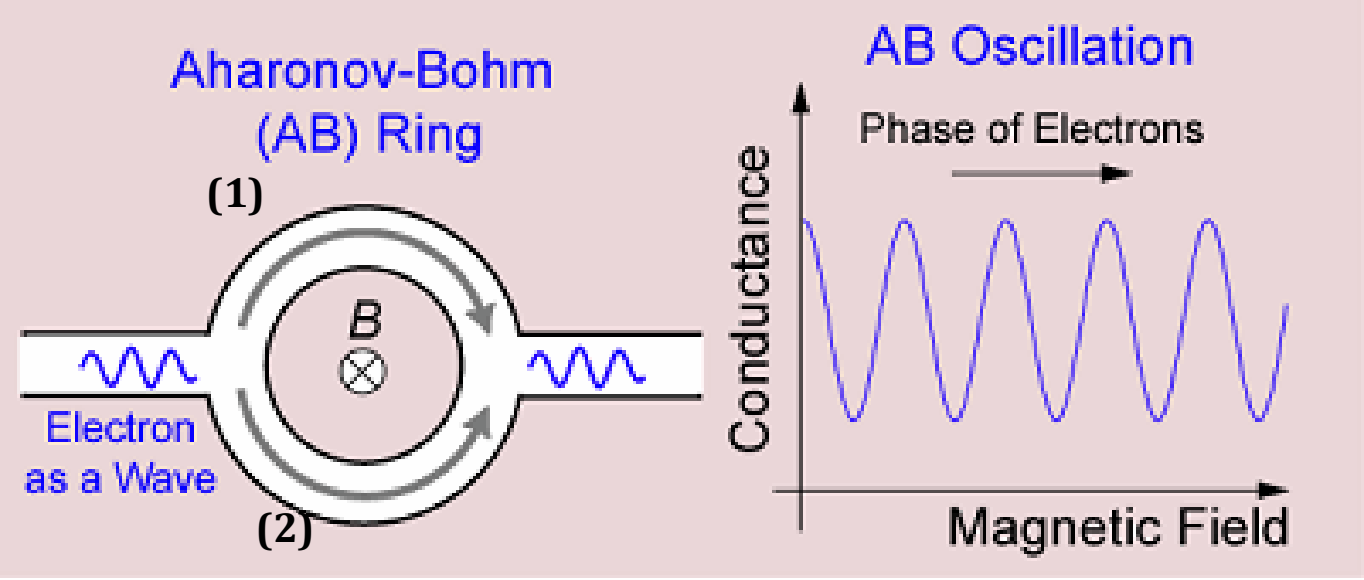
Weak Anti-Localization



In a system with the carrier's spin coupled to its momentum, the spin of the carrier rotates as it goes around a self-intersecting path, and the direction of this rotation is opposite for the two directions about the loop. Because of this, the two paths of any loop interfere *destructively*, which leads to a *lower* net resistivity. This is called *weak antilocalization*.

Aharonov-Bohm (A-B) Effect

An electron that enters the ring structure from the left has the option of two paths denoted as (1) and (2) in the figure. Interferences occur if the transport is phase conserving and if the phases of the two paths are shifted with respect to each other. The interference is measured as periodic oscillations of the current as function of the phase difference between the paths. The effect is named *Aharonov-Bohm effect*.



$$\Delta R = \frac{\pi r^2 B}{NWt2e}$$

The transport must be phase conserving. Hence the distance between the two contacts L must be smaller than the inelastic mean free path l_{in} and the phase coherence length l_Φ (diffusive transport)

$$l_e < L < l_{in}; l_F :$$

The phase difference between the two electron paths (1) and (2) of the ring is shifted by a magnetic flux. We consider the case of ballistic transport ($L < l_e$) where electrons propagate as plane waves of the form $\exp(i\boldsymbol{kr}) = \exp(i\boldsymbol{pr}/\hbar)$. In a magnetic field the momentum $\boldsymbol{p}$ in the Hamilton-operator is to be replaced by the kinetic momentum $\boldsymbol{p} - e\boldsymbol{A}$ with $\boldsymbol{A}$ the vector potential of the magnetic field. Hence the wave function of the electron in a magnetic field is that of a plane wave with modified momentum $\boldsymbol{p}$:

$$C.r:†/ \quad \exp\left(i\frac{e}{\hbar}A \cdot r\right) \exp.i.k \cdot r - x\ t††$$

The magnetic field causes a phase shift via $\boldsymbol{A}$, i.e.

$$d' \quad \hat{} \quad \frac{e}{\hbar}A \cdot ds:$$

The total phase shift is the integral of dφ along the path:

$$\Psi(\mathbf{r}) = \Psi_1 \exp\left(\frac{\mathrm{i}e}{\hbar}\int\limits_1 \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right) + \Psi_2 \exp\left(\frac{\mathrm{i}e}{\hbar}\int\limits_2 \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right)$$

Here Ψ_1 and Ψ_2 are the wave functions for vanishing magnetic field.

With respect to the magnetic field **B**, and thus with respect to the vector po- tential ***A*** the two paths are circulated in opposite direction and one obtains for the total phase shift:

$$\int\limits_1 \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} - \int\limits_2 \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \oint\limits_{\mathrm{Ring}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \iint \mathrm{d}\boldsymbol{f}\cdot\operatorname{curl}\boldsymbol{A} = \iint \mathrm{d}\boldsymbol{f}\cdot\boldsymbol{B} = \Phi_{\mathrm{B}}.$$

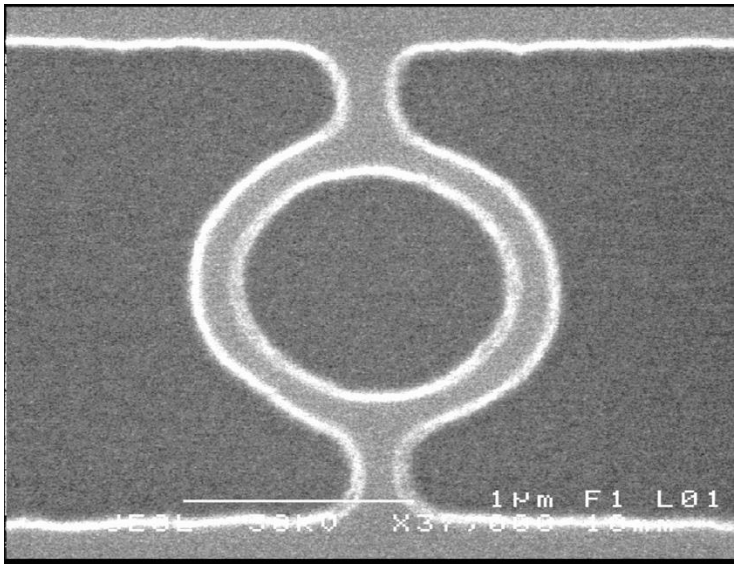
where Φ_{B} is the magnetic flux through the ring. Then,

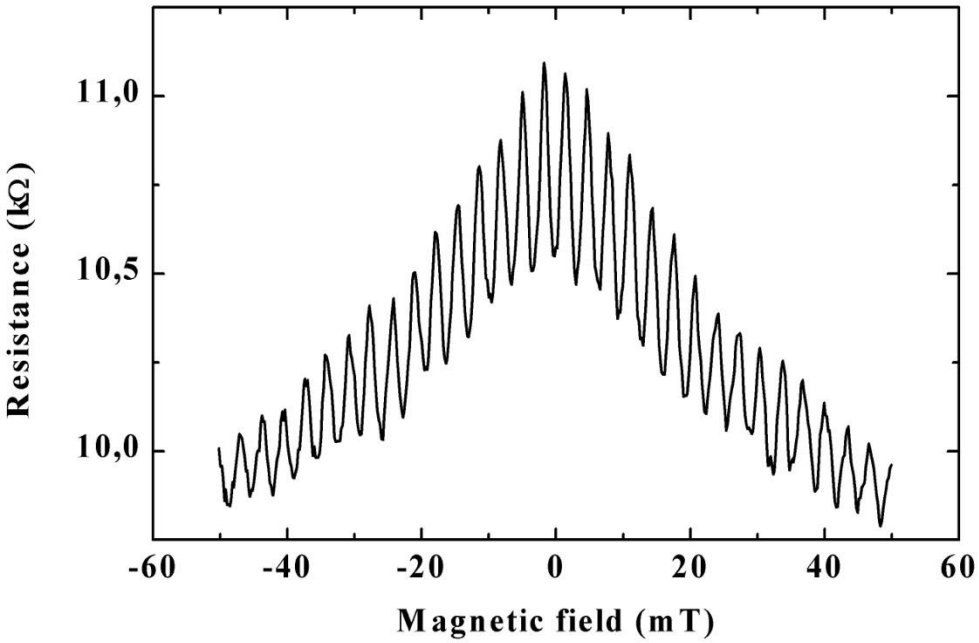
$$\Psi(\boldsymbol{r}) = \exp\left(\frac{\mathrm{i}e}{\hbar}\int\limits_2 \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right)\left(\Psi_1 \exp\left(\frac{\mathrm{i}e}{\hbar}\Phi_{\mathrm{B}}\right) + \Psi_2\right) \propto \mathrm{e}^{\frac{\mathrm{i}e}{\hbar}\Phi_{\mathrm{B}}} + \mathrm{e}^{\mathrm{i}\delta}.$$

Hence the current ***I*** as function of the magnetic flux Φ_{B} is

$$I \propto \left|\mathrm{e}^{\frac{\mathrm{i}e}{\hbar}\Phi_{\mathrm{B}}} + \mathrm{e}^{\mathrm{i}\delta}\right|^2 = 2[1+\cos(2\pi\Phi_{\mathrm{B}}/\Phi_0 - \delta)] \text{ , and } \Phi_0 = h/e \quad .$$

A-B Ring Applications

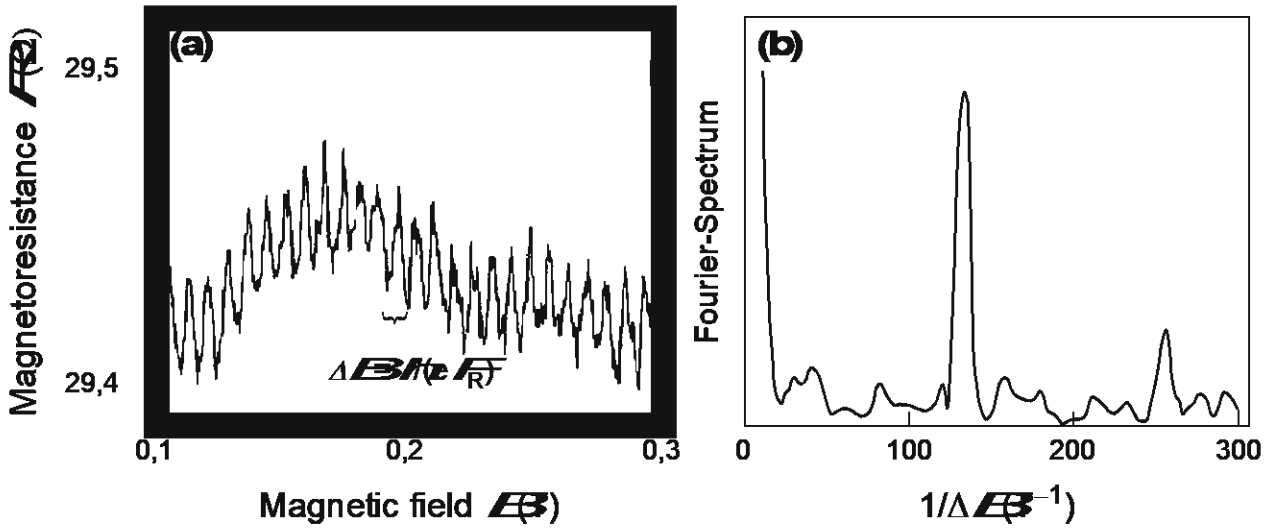




A-B Ring in Semiconductor

Altshuler-Aronov-Spivak (AAS) Oscillation

Aharonov-Bohm-oscillations as measured on a nanoscale gold ring:



Oscillations at twice the fundamental (~260 T^{-1}) originate in the interference of electrons at the entrance of the ring after travelling a full circle. Constructive interference at the entrance leads to a larger backscattering into the entrance channel and therefore to a reduction in the current. Such *Altshuler-Aronov-Spivak-oscillations* occur when the phase coherence length l_Φ is large enough. They are more robust than Aharonov-Bohm-oscillations since both partial waves travel the same path and thus encounter the same defect pattern.

Problems

1. Calculate the electric field dependence of the conductivity $\sigma(E)$ using the second iteration for the solution of the Boltzmann equation. In the second iteration step, as in the first, use a field-independent distribution f_0. Discuss interesting applications of a material with a highly field-dependent conductivity.

2. a) Calculate the density of states $D^{(1)}(E)$ for the electronic subband of a one-dimensional conductor. Sketch the density of states as function of the electron energy E for a sequence of subbands as they exist in a conducting wire.

b) Which scattering processes are possible in electron transport if only a single subband is occupied? What is different if a second subband is occupied so that the Fermi-level intersects two subbands?

3. Altshuler-Aronov-Spivak (AAS) oscillations occur because of the constructive interference at the entrance leads. Show that the oscillation period is now $\Phi_0/2$, half the period than for the Aharonov-Bohm (AB) effect.